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On surface biaxiality

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We propose the following: on the boundary of a nematic liquid crystal, the function which describes the molecular orientation is subject to a further symmetry condition, besides that reflecting apolarity. This additional symmetry delimits a class of biaxial states; within that class we study a model problem for a thin layer in which the anchoring energy prevails over the elastic energy in the presence of an electric field. We show that, when the anchoring and electric energies favour uniaxial states with optic axes at right angles, the equilibrium configuration migrates from one state to the other, traversing a whole family of biaxial states as the strength of the electric field increases.

1. Introduction

Since biaxial states were first observed in lyotropic nematic liquid crystals by Yu and Saupe [1], there has been an increasing interest in theories fit to describe their behaviour. Besides theories employing methods of theoretical physics, such as those by Saupe [2], and Govers and Vertogen [3], others were proposed in the style of continuum mechanics, such as that by Kini [4], recently emended by Leslie, Laverty and Carlsson [5].

Our analysis here proceeds along the lines followed by Ericksen [6]; we describe biaxial states through an order tensor which reflects on a macroscopic scale the essential features of the statistical distribution of the rod-like molecules. Such a description applies to biaxial as well as to uniaxial states; thus it is especially expedient in the study of changes from one phase to the other.

We recall that the order tensor, denoted here by \mathbf{M} is obtained as follows. Resort to the usual picture of an element of the liquid crystal, an element consisting of a great many molecules. Let f be the probability density of the molecular orientations: f is thus defined on the unit sphere S^2 . Then \mathbf{M} is the symmetric tensor of the second moments of f

$$\mathbf{M} := \int_{S^2} f(\mathbf{l}) \mathbf{l} \otimes \mathbf{l} \, da, \quad \mathbf{l} \in S^2. \quad (1.1)$$

Because we take the molecules to be apolar, f must be even

$$f(\mathbf{l}) = f(-\mathbf{l}), \quad \text{for all } \mathbf{l} \in S^2; \quad (1.2)$$

from this property and from the definition, it follows that \mathbf{M} has real non-negative proper numbers and unit trace, so it can be given the form

$$\mathbf{M} = \sum_{i=1}^3 \lambda_i \mathbf{n}_i \otimes \mathbf{n}_i, \quad \sum_{i=1}^3 \lambda_i = 1, \quad \lambda_i \geq 0, \quad (1.3)$$

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where λ_i are the proper numbers and \mathbf{n}_i the corresponding proper vectors. If, for some j , $\lambda_{j+1} = \lambda_{j+2}$, the sum of indices being defined *modulo* 3, then \mathbf{M} represents a uniaxial state, \mathbf{n}_j is the optic axis (denoted below by \mathbf{n}_E) and $s_E := \frac{1}{2}(3\lambda_j - 1)$ is the degree of orientation, as defined by Ericksen; \mathbf{M} reduces to

$$\mathbf{M} = s_E \mathbf{n}_E \otimes \mathbf{n}_E + \frac{1}{3}(1 - s_E) \mathbf{I}, \quad (1.4)$$

where \mathbf{I} is the identity tensor. When, further, s_E vanishes, \mathbf{M} becomes spherical: the liquid crystal *melts* and loses its optical anisotropy.

Here we do not consider the general class of \mathbf{M} shown by equation (1.3); rather, we restrict attention to a subclass, which includes, however, tensors \mathbf{M} as in equation (1.4). The restricted biaxiality that we envisage could, in our opinion, be observed with relative ease, if properly sought: it should be induced by boundary anchoring even in an ordinary nematic, the states of which are mostly uniaxial in the bulk.

To explain this idea, let us return to the usual picture of an element of liquid crystal placed at p . We suggest that, when p is immediately adjacent to the boundary of the region occupied by the material, the function f is forced to obey a further symmetry condition beyond (1.2), a condition which derives from the obstruction offered by the boundary.

Precisely, let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{v})$ be an orthogonal triad of unit vectors with \mathbf{v} along the normal to the boundary. Let the orientation \mathbf{l}_* be specified by

$$\mathbf{l}_* = \cos \theta \mathbf{v} + \sin \theta \cos \varphi \mathbf{e}_1 + \sin \theta \sin \varphi \mathbf{e}_2, \quad (1.5)$$

where $\theta \in [0, \pi[$ is the *tilt angle* and $\varphi \in [0, 2\pi[$ is the *twist angle*. We presume that the probability of finding a molecule along \mathbf{l}_* be equal to the probability of finding one along

$$\mathbf{l} = \cos \theta \mathbf{v} - \sin \theta \cos \varphi \mathbf{e}_1 - \sin \theta \sin \varphi \mathbf{e}_2; \quad (1.6)$$

\mathbf{l}^* is obtained from \mathbf{l} , through a rotation \mathbf{R}_v of angle π about \mathbf{v}

$$\mathbf{R}_v := 2\mathbf{v} \otimes \mathbf{v} - \mathbf{I}. \quad (1.7)$$

Thus, we formally require that f satisfy

$$f(\mathbf{R}_v \mathbf{l}) = f(\mathbf{l}) \quad \text{for all } \mathbf{l} \in S^2; \quad (1.8)$$

in other words, we extend to the projections of the peripheral molecules onto the plane tangent to the boundary the condition, valid in the bulk, of invariance under the reflection that exchanges the two ends of each molecule.

We prove in the next section that equation (1.8) makes \mathbf{v} a proper vector of \mathbf{M} ; thus we come to the special subclass, where one of the proper vectors \mathbf{n}_i in equation (1.3) is given *a priori*. Each element of the class is singled out by assigning the values of three parameters only, just as many as are needed to represent a uniaxial state. Thus, if the material is uniaxial in the bulk, but biaxial on the boundary, the same number of parameters suffices to describe it throughout the region it occupies, though different subclasses of tensors \mathbf{M} are involved at different points of the boundary, if the boundary is not plane.

If a surface bounding a nematic can induce biaxiality, we think that the model outlined here may be fit to explain how it does.

For a nematic liquid crystal uniaxial in the bulk, one would expect biaxiality to occur in a tiny layer all around the boundary, whereupon the tensor \mathbf{M} has at each point the normal as proper vector. Biaxial states are indeed rarely observed in the bulk.

The unquestionable evidence that they occur may solely rest on the observation of defects that are not compatible with uniaxial symmetry: observations of truly biaxial defects in a conventional thermotropic polymer are reported by De'neve, Kléman and Navard [7].

To simplify matters, here we study a problem where the liquid crystal occupies a thin layer between two infinite parallel plates, so that the direction of the outer normal \mathbf{v} be everywhere the same on the whole boundary. Thus, we can describe on both plates all biaxial states compatible with (1.8) through tensors \mathbf{M} having the proper vector \mathbf{v} in common. Using such a class of tensors throughout the whole layer, we can describe within it also the uniaxial states with optic axis either parallel or orthogonal to the bounding plates.

More precisely, let $(O, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ be a cartesian frame; consider the cell represented by

$$\mathcal{B} = \{(x, y, z) | x, y \in [0, l], z \in [0, d]\}, \tag{1.9}$$

so that the plates that bound \mathcal{B} are, respectively, at $z=0$ and $z=d$. On the former, following Nobili and Durand [8], we prescribe a weak anchoring, whose energy density is

$$\sigma_A := w_0 \operatorname{tr}((\mathbf{M} - \mathbf{M}_0)^2), \tag{1.10}$$

where w_0 is a positive constant and \mathbf{M}_0 is a given member in the special class of biaxial order tensors considered here. No anchoring condition is prescribed on the latter plate.

In the presence of an electric field, *modulo* an additive constant, we write the free energy density in the bulk as

$$\sigma_B := \kappa |\nabla \mathbf{M}|^2 + \sigma_0(\mathbf{M}) - \varepsilon_a \mathbf{E} \cdot \mathbf{M} \mathbf{E}, \tag{1.11}$$

where κ is an elastic constant, ε_a is the dielectric anisotropy (both positive), and σ_0 is a potential which we take as a function of the invariants of \mathbf{M} .

For simplicity, we assume that \mathbf{E} is constant throughout \mathcal{B} and that \mathbf{M} depends only on z . Thus, the total free energy functional per unit area reduces to

$$\mathcal{F}[\mathbf{M}] := \int_0^d \left(\kappa \left| \frac{d\mathbf{M}}{dz} \right|^2 + \sigma_0(\mathbf{M}) - \varepsilon_a \mathbf{E} \cdot \mathbf{M} \mathbf{E} \right) dz + w_0 \{ \operatorname{tr}(\mathbf{M} - \mathbf{M}_0)^2 \}_{z=0}. \tag{1.12}$$

To highlight the rôle played by the treated plate in inducing biaxiality, we consider a limiting problem which arises when d is very small.

Let $\xi \in [0, 1]$ be the variable defined by $z = \xi d$ and change $\mathbf{M}(z)$ into $\hat{\mathbf{M}}(\xi) = \mathbf{M}(\xi d)$; the functional \mathcal{F} becomes

$$\hat{\mathcal{F}}[\hat{\mathbf{M}}] := \int_0^1 \left(\kappa \left| \frac{d\hat{\mathbf{M}}}{d\xi} \right|^2 + d(\sigma_0(\hat{\mathbf{M}}) - \varepsilon_a \mathbf{E} \cdot \hat{\mathbf{M}} \mathbf{E}) \right) d\xi + w_0 \{ \operatorname{tr}(\hat{\mathbf{M}} - \mathbf{M}_0)^2 \}_{\xi=0}. \tag{1.13}$$

In the limit as $d \rightarrow 0^+$, $\hat{\mathcal{F}}$ may attain its minimum only if $\hat{\mathbf{M}}$ (and so \mathbf{M}) is constant within the layer, and so the functional in (1.12) reduces to an ordinary function. We neglect the potential σ_0 with respect to the electric energy density, as is appropriate for strong enough fields, since by equation (1.3) \mathbf{M} is bounded (and so is $\sigma_0(\mathbf{M})$). Thus we are left with the problem of minimizing

$$\mathcal{F}[\mathbf{M}] = w_0 \operatorname{tr}((\mathbf{M} - \mathbf{M}_0)^2) - \varepsilon_a d \mathbf{E} \cdot \mathbf{M} \mathbf{E}, \tag{1.14}$$

a problem that is completely solved in §3 below. We anticipate here only one conclusion of our study: suppose that \mathbf{M}_0 , the preferred state at the surface, be uniaxial

with the optic axis \mathbf{n}_E on the plane of the plate and that the electric field \mathbf{E} be on that plane also, but at right angles with \mathbf{n}_E ; the equilibrium configuration of the nematic will ultimately be uniaxial in the direction of \mathbf{E} as the strength of the field increases, but a whole family of biaxial states comes first *and* a singularity occurs at a critical field, when the order tensor that minimizes the energy represents a *conical* state, that is uniaxial as in (1.4) with a *negative* degree of orientation.

2. A simple class of biaxial distributions

2.1. *The tensor of second moments*

Here we explore the consequences of the additional symmetry (1.8) on the tensor \mathbf{M} in definition (1.1). Such a definition is equivalent to

$$\mathbf{M} = \int_{\mathbf{R}_v S^2} \mathbf{R}_v \mathbf{l} \otimes \mathbf{R}_v \mathbf{l} f(\mathbf{R}_v \mathbf{l}) \, da; \tag{2.1}$$

or, since both f and S^2 are invariant under \mathbf{R}_v , to

$$\mathbf{M} = \int_{S^2} \mathbf{R}_v \mathbf{l} \otimes \mathbf{R}_v \mathbf{l} f(\mathbf{l}) \, da. \tag{2.2}$$

Thus we find that \mathbf{R}_v and \mathbf{M} commute

$$\mathbf{R}_v \mathbf{M} = \mathbf{M} \mathbf{R}_v. \tag{2.3}$$

Applying either operator in equation (2.3) to the vector \mathbf{v} , we see that $\mathbf{M}\mathbf{v}$ is a proper vector of \mathbf{R}_v with proper value 1; hence, necessarily,

$$\mathbf{M}\mathbf{v} = m\mathbf{v}, \tag{2.4}$$

where m is an appropriate real number. We conclude that \mathbf{v} is a *proper vector of the tensor \mathbf{M}* .

Choosing again an orthonormal basis and standard spherical coordinates as in equation (1.5), we find that the components of \mathbf{M} are

$$\mathbf{M} = \begin{pmatrix} a & b & 0 \\ b & 1-a-s_v & 0 \\ 0 & 0 & s_v \end{pmatrix}, \tag{2.5}$$

where

$$\left. \begin{aligned} a &:= \int_{S^2} \sin^2 \theta \cos^2 \varphi f(\theta, \varphi) \, d\Omega, \\ b &:= \int_{S^2} \sin^2 \theta \cos \varphi \sin \varphi f(\theta, \varphi) \, d\Omega, \\ s_v &:= \int_{S^2} \cos^2 \theta f(\theta, \varphi) \, d\Omega. \end{aligned} \right\} \tag{2.6}$$

We must impose the condition that the proper numbers of \mathbf{M} be not negative. The characteristic polynomial of \mathbf{M} has the roots

$$\lambda_v = s_v, \quad \lambda_{\pm} = \frac{1-s_v \pm s_p}{2}, \tag{2.7}$$

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where we have introduced the parameter

$$s_p := \sqrt{[(2a - 1 + s_v)^2 + 4b^2]}. \tag{2.8}$$

Thus, equation (1.3)₃ implies that

$$0 \leq s_v \leq 1 \quad \text{and} \quad 0 \leq s_p \leq 1 - s_v. \tag{2.9}$$

In tensor form, \mathbf{M} can be written as

$$\mathbf{M} = \frac{3s_v - 1 + s_p}{2} \mathbf{v} \otimes \mathbf{v} + s_p \mathbf{n} \otimes \mathbf{n} + \frac{1 - s_v - s_p}{2} \mathbf{I}, \tag{2.10}$$

where $\mathbf{n} := (\cos \psi, \sin \psi, 0)$ is the proper vector of \mathbf{M} associated with λ_+ ; the angle ψ satisfies the equations

$$\begin{aligned} s_p \cos 2\psi &= 2a - (1 - s_v), \\ s_p \sin 2\psi &= 2b, \end{aligned} \tag{2.11}$$

which, when $s_p \neq 0$, determine ψ as follows:

$$\psi = \frac{1}{2} \arctg \frac{2b}{2a - 1 + s_v}. \tag{2.12}$$

Thus, \mathbf{M} is determined by three parameters, (a, b, s_v) or (s_v, s_p, ψ) , and so it can be represented by a point in a three-dimensional state space. From both conditions in (2.9), we easily see that in this space the admissible region is the cone C shown in figure 1. There the coordinate s_v is a measure of the *degree of tilt* of the molecules: the vertex $s_v = 1$ corresponds to a state completely ordered along \mathbf{v} , while on the plane $s_v = 0$ lie all states where the molecules are at right angles with the normal \mathbf{v} .

The second condition in (2.9) together with the definition (2.8) show that the intersection of every horizontal plane $s_v = \text{constant}$ with C is a circle of radius $\frac{1}{2}(1 - s_v)$

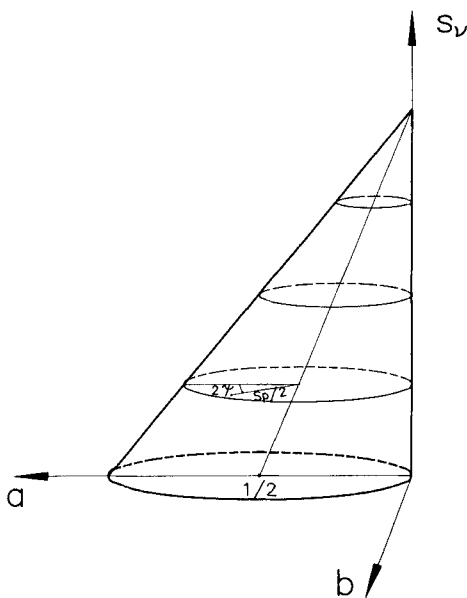


Figure 1.

centred in the point with $a = \frac{1}{2}(1 - s_v)$ and $b = 0$. Moreover, the distance of a point of this circle from the centre is equal to $\frac{1}{2}s_p$, while the angle between the plane (a, s_v) and the half-line from the centre through that point is 2ψ (see figure 1).

2.2. *The uniaxial states*

As already mentioned, uniaxial states are characterized by a tensor \mathbf{M} that has two equal proper numbers, and so they can be of three different families, which are illustrated in figure 2 and discussed in order below:

- (1) $\mathbf{n}_E = \mathbf{v}$ and $\lambda_- = \lambda_+$ (that is, $s_p = 0$). \mathbf{M} takes the form

$$\mathbf{M} = \frac{3s_v - 1}{2} \mathbf{v} \otimes \mathbf{v} + \frac{1 - s_v}{2} \mathbf{I} \tag{2.13}$$

and corresponds to uniaxial distributions with director along \mathbf{v} and any s_E in $[-\frac{1}{2}, 1]$. In the state space, they are represented by points on the axis of the cone C , the line that joins the centres of all the circles described above.

- (2) $\mathbf{n}_E = \mathbf{n}$ and $\lambda_v = \lambda_-$ (that is, $s_p = 1 - 3s_v$). Now \mathbf{M} corresponds to all uniaxial nematics that have a positive s_E and the director in the plane orthogonal to \mathbf{v} . Within C , they are represented by points on the surface of another cone which shares the base of C , and whose vertex is at the height $s_v = \frac{1}{3}$.
- (3) $\mathbf{n}_E = \mathbf{n}_\perp$ and $\lambda_v = \lambda_+$ (that is, $s_p = 3s_v - 1$). Here \mathbf{n}_\perp is the vector orthogonal to both \mathbf{v} and \mathbf{n} ; notice that s_E is now negative. Within C , the states are represented by points on the cone opposite to that where the members of family (2) lie.

In short, these families group all the uniaxial distributions in which the director is either the normal \mathbf{v} or a vector in the plane orthogonal to it. The fact that conical anchoring conditions, where the director makes an angle different from both 0 and $\pi/2$

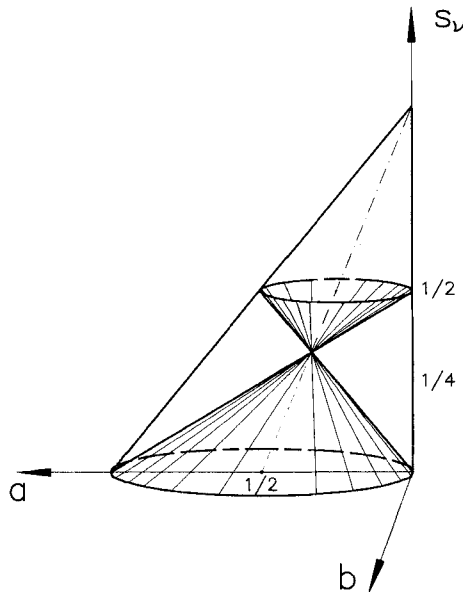


Figure 2.

with the normal, have seldom been observed in uniaxial nematics may support our requirement (1.8). If this requirement must hold, conical anchorings cannot even exist for uniaxial nematics, in the sense that they should be interpreted properly as biaxial states. (Here by *conical anchoring* we mean a boundary condition where only the angle between the director and the normal to the bounding surface is prescribed, and so the alignment is free to be along any generator of a cone.)

Finally, we note that the vertex common to both inner cones, i.e. the point such that $s_v = \frac{1}{3}$ and $s_p = 0$, represents the isotropic state with $\mathbf{M} = \frac{1}{3}\mathbf{I}$.

3. Equilibrium in the presence of an electric field

We will study now the effect of an electric field applied to a nematic whose biaxial states are represented by a tensor \mathbf{M} as in formula (2.10). We confine attention to fields applied either along the vector \mathbf{v} or in any direction orthogonal to it, because we want to preserve the symmetry which has led us to equation (2.10). We presume that the state cannot be described simply through equation (2.10), if \mathbf{E} is not directed as a proper vector of \mathbf{R}_v .

3.1. Planar electric field

We consider the problem of minimizing the free energy in (1.14) within the special class of order tensors (2.10) when also \mathbf{M}_0 , the order tensor preferred on the boundary, has been chosen in that class. In the frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{v})$, we write $\mathbf{n} = (\cos \psi, \sin \psi, 0)$ and take first the electric field \mathbf{E} as parallel to \mathbf{e}_1 .

Rescaling the function in equation (1.14) so as to make it dimensionless, we get

$$\mathcal{F}[\mathbf{M}] = \text{tr}((\mathbf{M} - \mathbf{M}_0)^2) - k\mathbf{E} \cdot \mathbf{M}\mathbf{E}, \tag{3.1}$$

where

$$\mathbf{M}_0 = \frac{3s_{v0} - 1 + s_{p0}}{2} \mathbf{v} \otimes \mathbf{v} + s_{p0} \mathbf{n}_0 \otimes \mathbf{n}_0 + \frac{1 - s_{v0} - s_{p0}}{2} \mathbf{I}, \tag{3.2}$$

with

$$\mathbf{n}_0 = (\cos \psi_0, \sin \psi_0, 0); \tag{3.3}$$

k is the positive constant

$$k := \frac{\epsilon_a d}{w_0}. \tag{3.4}$$

By elementary computations, equation (3.1) can be given the form

$$\mathcal{F}(s_v, s_p, \psi) = \frac{3}{2}(s_v - s_{v0})^2 + \frac{1}{2}(s_p^2 + s_{p0}^2) - s_p s_{p0} \cos 2(\psi - \psi_0) - \alpha(1 - s_v + s_p \cos 2\psi), \tag{3.5}$$

where α is given by

$$\alpha := \frac{kE^2}{2}. \tag{3.6}$$

We must find the absolute minimizer of \mathcal{F} in the set C of the admissible states.

The equilibrium equations in the interior of C are

$$\left. \begin{aligned} \frac{\partial \mathcal{F}}{\partial s_v} &= s_v - s_{v0} + \frac{\alpha}{3} = 0, \\ \frac{\partial \mathcal{F}}{\partial s_p} &= s_p - s_{p0} \cos 2(\psi - \psi_0) - \alpha \cos 2\psi = 0, \\ \frac{\partial \mathcal{F}}{\partial \psi} &= 2s_p[(\alpha + s_{p0} \cos 2\psi_0) \sin 2\psi - s_{p0} \sin 2\psi_0 \cos 2\psi] = 0, \end{aligned} \right\} \quad (3.7)$$

and so the equilibrium values of s_v , s_p and ψ are, respectively,

$$\left. \begin{aligned} s_v^{EQ}(\alpha) &= s_{v0} - \frac{\alpha}{3}, \\ s_p^{EQ}(\alpha) &= \sqrt{[s_{p0}^2 \sin^2 2\psi_0 + (\alpha + s_{p0} \cos 2\psi_0)^2]}, \\ \psi^{EQ}(\alpha) &= \frac{1}{2} \operatorname{arctg} \frac{s_{p0} \sin 2\psi_0}{\alpha + s_{p0} \cos 2\psi_0}. \end{aligned} \right\} \quad (3.8)$$

The main features of these solutions for different values of s_{v0} , s_{p0} , and ψ_0 , as they occur with increasing α , are as follows. Clearly, for $\alpha=0$ the equilibrium state is represented in the cone by the point (s_{v0}, s_{p0}, ψ_0) , whence for $\alpha > 0$ it starts a trajectory of equilibrium solutions described by equations (3.8).

The first equation in (3.8) tells us that the system is driven towards the plane (a, b) as α increases; the second and the third remain valid only as long as the point $(s_v^{EQ}, s_p^{EQ}, \psi^{EQ})$ remains inside the cone. For large values of α , the evolution is different depending on whether the equilibrium point hits first the base or the lateral surface of the cone.

In the former case, the state evolves on the plane and the complete description of the trajectory of the equilibrium point is given by

$$\left. \begin{aligned} s_v^{EQ}(\alpha) &= \max \{s_{v0} - \alpha/3, 0\}, \\ s_p^{EQ}(\alpha) &= \min \{\sqrt{[s_{p0}^2 \sin^2 2\psi_0 + (\alpha + s_{p0} \cos 2\psi_0)^2]}, 1\}, \\ \psi^{EQ}(\alpha) &= \frac{1}{2} \operatorname{arctg} \frac{s_{p0} \sin 2\psi_0}{\alpha + s_{p0} \cos 2\psi_0}. \end{aligned} \right\} \quad (3.9)$$

In the latter case, the constrained minimum problem for \mathcal{F} on the cone has a slightly different solution: $\psi^{EQ}(\alpha)$ is the same as before, but

$$\left. \begin{aligned} s_v^{EQ}(\alpha) &= \max \left\{ \frac{1}{2} [s_{v0} - \alpha/3 + 1 - \sqrt{[s_{p0}^2 \sin^2 2\psi_0 + (\alpha + s_{p0} \cos 2\psi_0)^2]}], 0 \right\}, \\ s_p^{EQ}(\alpha) &= 1 - s_v^{EQ}(\alpha), \end{aligned} \right\} \quad (3.10)$$

for every $\alpha \geq \tilde{\alpha}_0$, where $\tilde{\alpha}_0$ is the value of α for which the equilibrium point in equations (3.8) reaches the lateral surface.

Some important features of these solutions deserve notice:

- (1) For any starting point (s_{v0}, s_{p0}, ψ_0) , the equilibrium trajectory remains in the plane $b = \text{constant}$ until it reaches the circumference on the base of the cone, as can be easily checked from the equations

$$b^{EQ}(\alpha) = \frac{1}{2} s_p^{EQ}(\alpha) \sin 2\psi^{EQ}(\alpha) = \frac{1}{2} s_{p0} \sin 2\psi_0. \quad (3.11)$$

Moreover, the trajectory of the equilibrium point on that plane is the line $2s_v + a = \text{constant}$, as long as it remains in the interior of the cone.

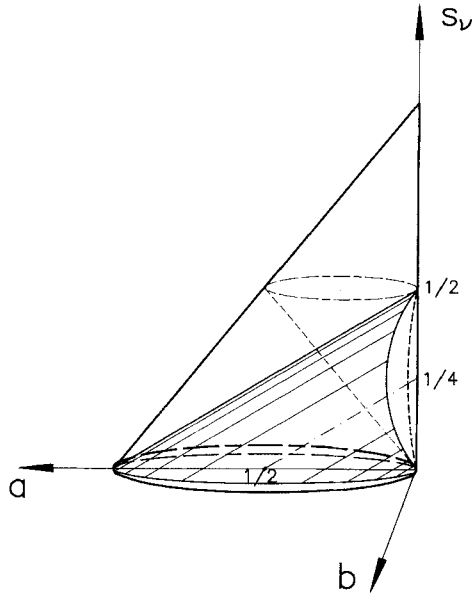


Figure 3.

- (2) There are two regions within the cone whence two different types of equilibrium trajectories start. Those starting from the one region hit the lateral surface first, while those starting from the other hit the base of the cone first. These two regions are adjacent and separated by the surface represented by the equations

$$9s_v^2 + 6s_v s_p \cos 2\psi + s_p^2 = 1. \tag{3.12}$$

In the variables (a, b, s_v) such an equation becomes simply

$$(4s_v + 2a - 1)^2 + 4b^2 = 1, \tag{3.13}$$

and represents a cylinder of radius $\frac{1}{2}$, with axis along the line

$$\left. \begin{aligned} b &= 0, \\ 4s_v + 2a &= 1. \end{aligned} \right\} \tag{3.14}$$

This cylinder (see figure 3); is tangent to the two inner cones shown in figure 2, and leaves the lower one entirely under it and the upper one entirely above.

- (3) Among the equilibrium trajectories that start from a uniaxial state, only those with optic axis \mathbf{n}_E parallel to the electric field remain uniaxial for all values of the applied field. A peculiar transition takes place if the initial state is uniaxial with \mathbf{n}_E on the plane, at right angles with the applied field, and $s_E > 0$. All the equilibrium trajectories starting from these points, which lie on the intersection of the lower inner cone with the plane $b = 0$, are characterized by a discontinuity in the angle ψ , which jumps from $\pi/2$ to 0 when the trajectory intercepts the line $s_p = 0$, i.e. when

$$\alpha = s_{E0} := 1 - 3s_{v0}. \tag{3.15}$$

3.2. Normal electric field

When $\mathbf{E} = E\mathbf{v}$, the solution is much easier to find: the free energy becomes

$$\mathcal{F}(s_v, s_p, \psi) = \frac{3}{2}(s_v - s_{v0})^2 + \frac{1}{2}(s_p^2 + s_{p0}^2) - s_p s_{p0} \cos 2(\psi - \psi_0) - 2\alpha s_v, \tag{3.16}$$

and possesses the following equilibrium configurations

$$\left. \begin{aligned} s_v^{EQ}(\alpha) &= \begin{cases} s_{v0} + \frac{2}{3}\alpha & \text{if } \alpha \leq \tilde{\alpha}_1, \\ \frac{1}{4}(3s_{v0} + 2\alpha + 1 - s_{p0}) & \text{if } \tilde{\alpha}_1 \leq \alpha \leq \tilde{\alpha}_2, \\ 1 & \text{if } \alpha \geq \tilde{\alpha}_2, \end{cases} \\ s_p^{EQ}(\alpha) &= \begin{cases} s_{p0} & \text{if } \alpha \leq \tilde{\alpha}_1, \\ 1 - s_v^{EQ}(\alpha) & \text{if } \alpha \geq \tilde{\alpha}_1, \end{cases} \\ \psi^{EQ}(\alpha) &= \psi_0, \end{aligned} \right\} \tag{3.17}$$

where

$$\tilde{\alpha}_1 := \frac{3}{2}(1 - s_{v0} - s_{p0}) \quad \text{and} \quad \tilde{\alpha}_2 := \frac{1}{2}(3 - 3s_{v0} + s_{p0}). \tag{3.18}$$

The main feature of these solutions is that every trajectory in the cone C reaches the vertex at $\alpha = \tilde{\alpha}_2$, in contrast with the preceding case, when only the trajectories starting from states with ψ_0 equal to 0 or $\pi/2$ succeed in reaching the point with $s_v = 0$ and $\psi = 0$ for a finite value of α .

4. Conclusions

From the results of our analysis we draw the following qualitative conclusions, perhaps of interest for the experimentalist.

We have studied the effect of an electric field \mathbf{E} on a thin layer of nematic liquid crystal resting on a plate treated so as to induce a preferred uniaxial state with optic axis \mathbf{n}_E parallel to the layer and degree of orientation $s_E > 0$.

When \mathbf{E} and \mathbf{n}_E are parallel, the state of the liquid crystal remains uniaxial for all values of the strength of \mathbf{E} . When \mathbf{E} is not parallel to \mathbf{n}_E , but still lies in the plane of the layer, it affects the molecular alignment in two different ways: the induced change depends on whether the angle ψ_0 between \mathbf{E} and \mathbf{n}_E is greater than $\pi/4$ or not. In both cases the state becomes biaxial as soon as the field is applied.

Suppose that $0 < \psi_0 < \pi/4$. As the strength E of the electric field increases, one proper vector of the molecular order tensor \mathbf{M} in the plane of the layer tends to become parallel to \mathbf{E} , while the scalar order parameters s_p and s_v defined above exhibit opposite behaviour: the former increases while the latter decreases. If $\pi/4 < \psi_0 < \pi/2$, the effect of increasing E is as before on s_v ; however, now s_p first decreases and then increases, *but never vanishes*. Finally, if $\psi_0 = \pi/2$, both s_p and s_v behave as before, but now the former vanishes for a critical value of E for which, in addition, the proper vectors of \mathbf{M} in the plane of the layer suffer a sudden rotation of angle $\pi/2$ about the axis normal to the layer.

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